

ON THE VISUAL APPEARANCE OF RELATIVISTIC OBJECTS

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ABSTRACT. It has been almost half a century since the realization [1,3] that an object moving at relativistic speeds (and observed by light reflected from some point source) is seen not as squashed (as a naive interpretation of the Lorentz-Fitzgerald contraction would suggest) but rather as **rotated**, through an angle dependent on its velocity and direction of motion. I will try to show here that this subject continues to be worth exploring.

The author asserts his moral right to remuneration for any applications of these ideas to computer games or big-budget sci-fi movies.

1 There is a detailed account of this subject in Taylor's textbook [2], which I will take as point of departure; but it will simplify some things to use notation slightly different from his, as follows: We observe an object moving across our field of view at constant velocity v , illuminated by some fixed point light source. Choose coordinates in which we observers are situated at the origin, and such that the path of the observed object is a line in the upper half of the (x, y) -plane, parallel to the x -axis (as in Taylor's Fig. X-5 p. 352). Let $\tilde{\psi}$ denote the angle between our line of sight to the object, and its path, normalized so that $\tilde{\psi} = 0$ when the object crosses the y -axis; thus

$$\tilde{\psi} = \psi - \pi/2 ,$$

where ψ is the angle of observation used in Taylor's calculations.

A relativistic ray-tracing argument now shows that the observed object will appear to us as rotated counterclockwise through an angle ϕ , satisfying the equation [Taylor X-12 p. 356]

$$\cos(\phi + \psi) = \frac{\cos \psi - v}{1 - v \cos \psi}$$

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(with the velocity v measured in units such that the speed of light $c = 1$).

2 Let

$$\lambda : S^1 = \mathbb{R}/2\pi\mathbb{Z} \rightarrow [-\infty, +\infty]$$

denote the function $x \mapsto \operatorname{arctanh} \sin x$: this is the inverse to the function studied by C. Gudermann (1798 - 1852). It has many other representations, eg

$$\lambda(x) = \log |\tan x + \sec x| = \frac{1}{2} \log \left| \frac{1 + \sin x}{1 - \sin x} \right| ,$$

and is antiperiodic: $\lambda(x + \pi) = -\lambda(x)$. It will often be convenient here to regard it as defined on the interval $[-\pi, +\pi]$. Its derivative, as industrious students of elementary calculus learn, is $\sec x$, from which the power series representation

$$\lambda(x) = \sum_{n \geq 0} (-1)^n E_n \frac{x^{2n+1}}{(2n+1)!}$$

is easily deduced. [The Euler numbers E_n are entry A000364 in Sloane's online database of integer sequences.]

Definition The two-variable formal power series

$$X +_M Y := \lambda^{-1}(\lambda(X) + \lambda(Y)) = X + Y + \cdots \in \mathbb{Q}[[X, Y]]$$

is a one-dimensional formal group law, with λ as its logarithm.

This group law is closely related to Mercator's projection, discussed below. If we write

$$\nu = \arcsin v ,$$

then we can state the

Proposition: The observed rotation ϕ satisfies the equation

$$\phi + \tilde{\psi} = \nu +_M \tilde{\psi} .$$

PROOF: We can restate Taylor's formula as

$$\cos(\phi + \tilde{\psi} + \pi/2) = \frac{\cos(\tilde{\psi} + \pi/2) - \sin \nu}{1 - \sin \nu \cos(\tilde{\psi} + \pi/2)} ,$$

ie as

$$\sin(\phi + \tilde{\psi}) = \frac{\sin \tilde{\psi} + \sin \nu}{1 + \sin \nu \sin \tilde{\psi}} .$$

On the other hand, the proposition asserts that

$$\lambda(\phi + \tilde{\psi}) = \operatorname{arctanh} \sin(\phi + \tilde{\psi})$$

equals

$$\lambda(\phi) + \lambda(\nu) = \operatorname{arctanh} \sin \tilde{\psi} + \operatorname{arctanh} \sin \nu .$$

Taking hyperbolic tangent of both sides, and using the addition formula for that function, gives

$$\sin(\phi + \tilde{\psi}) = \tanh \left[\operatorname{arctanh} \sin \tilde{\psi} + \operatorname{arctanh} v \right] = \frac{\sin \tilde{\psi} + v}{1 + v \sin \tilde{\psi}}$$

as claimed.

3 Mercator's projection sends a point on the sphere with latitude ϕ to a point in the plane with y -coordinate $\lambda(\phi)$; this is essentially just the logarithm of stereographic projection. The formal group law defined above thus combines the line-of-sight angle $\tilde{\psi}$ and the relativistic velocity angle ν by sending them separately to their Mercator projections to the line, adds them as real numbers, and converts their sum back to an angle. In particular, when $\tilde{\psi} = 0$ (the moving object is at its closest approach) the object is seen as rotated through the angle $\arcsin v$. That relativistic geometry is somehow a complex (Wick) rotation of Euclidean geometry is well-known, but this seems to be a very explicit form of that correspondence.

It is easy to check that the derivative of

$$\lambda^{-1}(x) = \arcsin \tanh x$$

is the hyperbolic secant. This suggests that, remarkably enough,

$$\lambda^{-1}(x) = -i\lambda(ix) ,$$

which is also easily verified. In other words, if we write $\Lambda(x) = \lambda(ix)$, then

$$(\Lambda \circ \Lambda)(X) = -X$$

as formal power series.

Now $\sin x = j(\exp(ix))$ with $j(z) = \frac{1}{2i}(z - z^{-1})$ and $\tanh x = k(\exp(2x))$ with

$$k(z) = \frac{z-1}{z+1} = \begin{bmatrix} 1 & -1 \\ 1 & +1 \end{bmatrix} (z) ,$$

so $\lambda(x) = \frac{1}{2} \log(k^{-1} \circ j)(\exp(ix)) = \log([M](\exp(ix)))$, where $[M] \in \operatorname{PGL}_2(\mathbb{C})$ is the element of order four represented by

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -i \\ 1 & +i \end{bmatrix} .$$

The matrix on the right represents the Cayley transform

$$z \mapsto C(z) = \frac{z - i}{z + i}$$

which, on the unit circle, is essentially just stereographic projection onto the imaginary axis; thus

$$\lambda(x) = -\log iC(e^{ix}) .$$

It follows that

$$X +_M Y \equiv i \log \frac{\cos \frac{1}{2}(X - Y) - i \sin \frac{1}{2}(X + Y)}{\cos \frac{1}{2}(X - Y) + i \sin \frac{1}{2}(X + Y)} \pmod{2\pi}$$

extends to a continuous map

$$S^1 \times S^1 - \{(\pm \frac{1}{2}\pi, \mp \frac{1}{2}\pi)\} \ni (X, Y) \mapsto X +_M Y \in S^1$$

from the twice-punctured torus to the circle.

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REFERENCES

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